A relation between prime numbers of the form 4k + 1 and odd numbers, sum of two squares

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Let's consider $\pi(x)$, the number of prime numbers smaller or equal to x. Excluding number 2, prime numbers can be divided in those that can be written in the form 4k + 1 and those that can be written in the form 4k + 3, where k is a natural number.

We define $\pi_{4k+1}(x)$ as the number of prime numbers smaller or equal to x of the form 4k + 1, while $\pi_{4k+3}(x)$ is the number of prime numbers smaller or equal to x of the form 4k + 3. Then

$$\pi(x) = \pi_{4k+1}(x) + \pi_{4k+3}(x) + 1$$

From the prime number theorem for arithmetic progressions, we know that when $x \to \infty$,

$$\pi_{4k+1}(x) / \pi_{4k+3}(x) \to 1.$$

Note that for values of $x < 10^{11}$, $\pi_{4k+1}(x)$ is almost always smaller than $\pi_{4k+3}(x)$: $\pi_{4k+1}(x)$ is greater than $\pi_{4k+3}(x)$ only in less than 2.6 % of the cases.[2]

In this work, we propose an astonishing relation between $\pi_{4k+1}(x)$ and the distribution of odd numbers resulting of the sum of two square numbers.

Let's consider the odd numbers

$$t = (2r)^2 + (2s - 1)^2 \qquad (1)$$

where r and s are natural numbers : 1, 2, 3, 4…

Now we define the counting functions :

- M(x), the number of t's smaller or equal to x.

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2 Andrew Granville and Greg Martin, "Prime Number Races", *The American Monthly*, vol. 113, n° 1 (2006), http://www.dms.umontreal.ca/~andrew/PDF/PrimeRace.pdf.

In very good approximation : $M(x) = (\pi/16)x - (1/4)x^{1/2}$.

- N(x), the number of t's smaller or equal to x, excluding repetitions.[3]
- P(x), the number of t's smaller or equal to x, that appear with frequency one.

For example, 13 = 4 + 9 has frequency 1, and is counted in the three functions for $x \geq 13$, while 65 = 16 + 49 = 64 + 1 has frequency 2, and adds 2 to M(x), 1 to N(x), and 0 to P(x), for $x \geq 65$. Table 1A shows the values of the counting functions for different powers of 10. Euler has shown all prime numbers of the forms 4k + 1 are contained in P(x).

Next, let's define the following two functions:

$$n(x) = (2/\ln x)^{1/2} N(x),$$

$$p(x) = (2/\pi)^{1/2} P(x),$$

from Table 2A, we can see that, for $x \leq 10^9$,

$[n(x) + p(x)]/2$ is practically equal to $\pi_{4k+1}(x)$.

In Table 3A we take the ratio of these two quantities to make clearer this similarity.

It remains to see if "practically equal" holds for $x > 10^9$.

We do the same exercise redefining the counting functions so that r and s in t can also take the value of 0.

Defining now

$$t' = (2r)^2 + (2s + 1)^2 \qquad (2)$$

with r and s beginning from zero: 0, 1, 2, 3, 4…

The corresponding numbers for M'(x), N'(x), P'(x), n'(x), p'(x) are provided in Tables 1B, 2B and 3B. Now, we have, in excellent approximation, $M'(x) = (\pi/16)x + (1/4)x^{1/2}$.

From Table 4, where we take the difference of $[(n'(x) + p'(x)] / 2$ and $[(n(x) + p(x)] / 2$, we see that as this differences is relatively small, we can take any set of counting functions, both show a striking similarity with $\pi_{4k+1}(x)$.

3 For more details, Erika Lorena Álvarez Ramirez and Jean Pestieau, *Note sur la distribution des nombres premiers*, décembre 2013, http://www.d-meeus.be/math/2013-12-18nombrespremiers.pdf.

Table 1A

x	M(x)	N(x)	P(x)
10^2	17	15	13
10^3	188	143	103
10^4	1939	1280	777
10^5	19559	11498	5993
10^6	196093	104761	48360
10^7	1962716	967734	405406
10^8	19632440	9032945	3487280
10^9	196341632	85014042	30605537

Table 2A

x	n(x)	p(x)	[n(x) + p(x)] / 2	$\pi_{4k+1}(x)$	π(x) / 2
10^2	9.89	10.37	10.13	11	12.5
10^3	76.95	82.18	79.56	80	84.0
10^4	596.47	619.96	608.21	609	614.5
10^5	4792.31	4781.72	4787.01	4783	4796.0
10^6	39859.44	38585.70	39222.57	39175	39249.0
10^7	340889.90	323467.19	332178.54	332180	332289.5
10^8	2976401.58	2782446.87	2879424.22	2880504	2880727.5
10^9	26410496.39	24419685.45	25415090.92	25423491	25423767.0

Table 3A

x	π(x)	$2\pi_{4k+1}(x)$ / [n(x) + p(x)]	π(x) / [n(x) + p(x)]
10^2	25	1.086	1.234
10^3	168	1.006	1.056
10^4	1229	1.0013	1.0103
10^5	9592	0.99916	1.00188
10^6	78498	0.998787	1.000674
10^7	664579	1.000004	1.000334
10^8	5761455	1.000375	1.000453
10^9	50847534	1.000331	1.000341

Table 1B

x	M'(x)	N'(x)	P'(x)
10^2	22	19	16
10^3	204	153	108
10^4	1989	1306	783
10^5	19717	11568	5993
10^6	196593	104962	48333
10^7	1964297	968304	405234
10^8	19637440	9034625	3486636
10^9	196357443	85019033	30603269

Table 2B

x	n'(x)	p'(x)	[n'(x) + p'(x)] / 2	$\pi_{4k+1}(x)$	π(x)/2
10^2	12.52	12.77	12.64	11	12.5
10^3	82.33	86.17	84.25	80	84.0
10^4	608.58	624.74	616.66	609	614.5
10^5	4821.48	4781.72	4801.60	4783	4796.0
10^6	39935.91	38564.15	39250.03	39175	39249.0
10^7	341090.69	323329.95	332210.32	332180	332289.5
10^8	2976955.15	2781933.03	2879444.09	2880504	2880727.5
10^9	26412049.90	24417875.85	25414961.37	25423491	25423767.0

Table 3B

x	$\pi(x)$	$2\pi_{4k+1}(x) / [n'(x) + p'(x)]$	$\pi(x) / [n'(x) + p'(x)]$
10^2	25	0.870	0.989
10^3	168	0.950	0.997
10^4	1229	0.9876	0.9965
10^5	9592	0.99613	0.99883
10^6	78498	0.998088	0.999974
10^7	664579	0.999909	1.000238
10^8	5761455	1.000368	1.000446
10^9	50847534	1.000336	1.000346

Table 4

x	$[(n'(x)+p'(x)] / 2 - [(n(x)+p(x)] / 2$
10^2	2.51
10^3	4.69
10^4	8.45
10^5	14.59
10^6	27.46
10^7	31.78
10^8	19.87
10^9	− 129.55